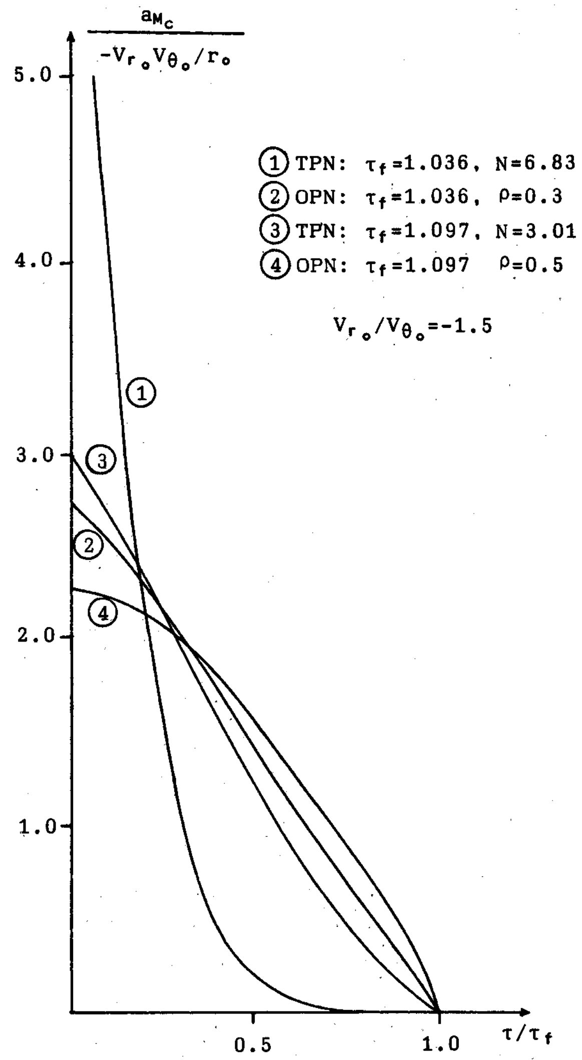

Fig. 3 The time histories of commanded missile acceleration for OPN and TPN.

As mentioned previously, t is a monotonically increasing function of θ and vice versa, provided $\dot{\theta}_0$ is positive. The time of capture is obtained by letting $\theta = \theta_f$ in Eq. (15).

Numerical Results

Simulation has been employed to demonstrate the superiority of optimal proportional navigation (OPN). A comparison was made with TPN in which the commanded missile acceleration is given by

$$a_{M_c} = -NV_{r_0}\dot{\theta} \tag{16}$$

and N is the navigation constant.

For comparison, a nonmaneuvering target was considered. The time of capture is depicted in Fig. 2, where the curve for TPN is obtained by the formula[4]

$$\tau_f = \frac{-t_f}{r_0/V_{r_0}} = 1 - \frac{1}{1 + (V_{r_0}/V_{\theta_0})^2\,(1-2N)} \tag{17}$$

and the corresponding τ_f for OPN can be found by the use of Eq. 15. It is seen that over most of the range of ρ and N, which are compatible with the capture-point condition, the time of capture of OPN is lower than that of TPN. Especially, when $(V_{r_0}/V_{\theta_0})^2\,(1-2N)$ is close to -1, TPN requires a time that is almost several times as long as that required by OPG. The lowest posible τ_f that can be attained by TPN is equal to 1, but the τ_f can be reduced arbitrarily near 0 by OPG.

To compare the acceleration properties, we keep the time of capture the same in both the laws by the special choices of ρ and N, and the resulting time histories of commanded missile accelerations are depicted in Fig. 3. It can be observed that maximum required acceleration for OPN control is reduced largely from the TPN requirement.

Conclusion

The problem of finding a nonlinear optimal guidance law for a homing missile with commanded acceleration applied normal to the LOS so as to capture a maneuvering target with a predetermined trajectory, while minimizing a weighted linear combination of time of capture and energy expenditure, has been solved in closed form. The derived optimal control law is equal to the LOS rate multiplied by a cosine function of aspect angle, which can be implemented easily. From the numerical simulation, the resulting guidance law appears to yield a significant advantage over true proportional navigation.

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Dynamics and Control of a Space Platform with a Tethered Subsatellite

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Introduction

THE analyses of the dynamics and control of the tethered subsatellite system (TSS) have been performed by a host of investigators. It was noted that for local vertical stationkeeping, within the linear range, tether tension would not provide control of the out-of-orbit-plane swing motion, but such control would be implemented in the nonlinear system due to higher-order coupling, or by including nonlinear feedback terms in the tension-control law. Bainum and Kumar[1]

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first introduced a new tether tension control law based on linear optimal control theory applicable for the TSS. Subsequently, a mathematical model for the tethered platform system (TPS),[2] restricted to the orbital plane, was developed. The objective of the present paper is the development of the three-dimensional mathematical model for a space platform-based application of the tethered subsatellite and the synthesis of appropriate control laws based on an application of optimal linear regulator control theory; the out-of-plane swing motion could be controlled by a momentum-type controller on the platform only. This represents the expansion of the two-dimensional model of Ref. 2.

Development of the System Equations of Motion

The system is idealized as containing a thin rigid platform from which an assumed (massless) tether is deploying or retrieving a subsatellite at a distance ℓ from a point on the platform that is offset by distances h_1 and h_2 along the roll and pitch axes, respectively, from the mass center of the platform (Fig. 1). The tether is considered to remain taut (rigid) for all subsatellite motion. For this study the mass of the subsatellite is assumed to be significantly less than the mass of the platform. The platform center of mass is assumed to follow a circular orbit. Environmental disturbances are neglected.

The position vector describing the location of the subsatellite is

$$\begin{aligned}\bar{R} &= \bar{R}_o + \bar{\ell}\\ &= [R_o(c\theta s\phi s\psi - s\theta c\psi) + h_1 + \ell s\alpha c\gamma]\hat{e}_\xi\\ &\quad + [R_o(c\theta s\phi c\psi + s\theta s\psi) + h_2 + \ell s\gamma]\hat{e}_\eta\\ &\quad + [R_o c\theta c\phi + \ell c\alpha c\gamma]\hat{e}_\zeta \end{aligned} \tag{1}$$

where R_o represents the distance between the center of the Earth and the platform center of mass 0, and ℓ represents the length of the tether line. θ, ϕ, ψ, α, γ represent the platform angular displacements (pitch, roll, yaw), the tether in-plane swing, and out-of-plane swing angles, respectively. The body axes $\hat{e}_\xi$, $\hat{e}_\eta$, $\hat{e}_\zeta$ coincide with the platform principal axes of inertia $s \to \sin(\ )$ and $c \to \cos(\ )$. An Euler angle sequence is assumed in the order of 1) pitch (θ), 2) roll (ϕ), and 3) yaw (ψ), starting from the orbit (local vertical/horizontal) frame and ending with the platform principal axes.

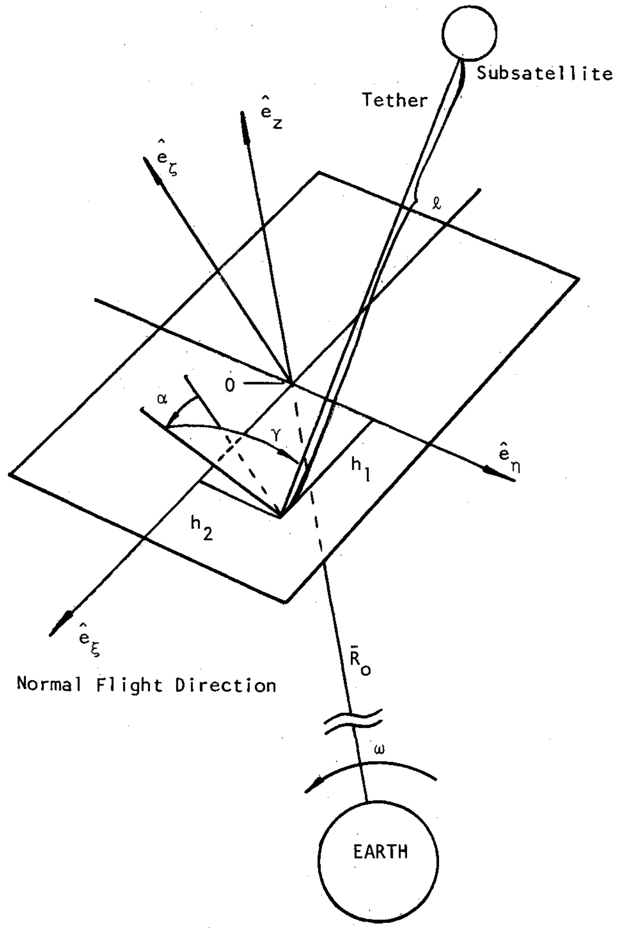

Fig. 1 System geometry.

The kinetic energy and the potential energy take the form

$$\begin{aligned} T &= T_P + T_S\\ &= (½)MR_o^2\omega^2 + (½)(I_\xi\Omega_\xi^2 + I_\eta\Omega_\eta^2 + I_\zeta\Omega_\zeta^2) + (½)m\dot{\bar{R}}\cdot\dot{\bar{R}}\\ &= (½)MR_o^2\omega^2 + (½)\{I_\xi[\dot{\phi}^2c^2\psi + \dot{\phi}(\dot{\theta}+\omega)c\phi s2\psi\\ &\quad + (\dot{\theta}+\omega)^2c^2\phi s^2\psi] + I_\eta[\dot{\phi}^2s^2\psi - \dot{\phi}(\dot{\theta}+\omega)c\phi s2\psi\\ &\quad + (\dot{\theta}+\omega)^2c^2\phi c^2\psi] + I_\zeta[\dot{\psi}^2 - 2\dot{\psi}(\dot{\theta}+\omega)s\phi\\ &\quad + (\dot{\theta}+\omega)^2s^2\phi]\} + (½)m\dot{\bar{R}}\cdot\dot{\bar{R}} \end{aligned} \tag{2}$$

where $\dot{\bar{R}}$ is obtained from Eq. (1).

$$\begin{aligned} V &= V_P + V_S\\ &= -\int(GM_o/r)\mathrm{d}m_P - GM_o m/|\bar{R}|\\ &= (GM_o/4R_o^3)\{I_\xi[3(s^2\theta - c^2\theta s^2\phi)c2\psi - 3s2\theta s2\psi s\phi\\ &\quad - 3c^2\theta c^2\phi + 1] + I_\eta[3(c^2\theta s^2\phi - s^2\theta)c2\psi + 3s2\theta s\phi s2\psi\\ &\quad - 3c^2\theta c^2\phi + 1] + I_\zeta[6c^2\theta c^2\phi - 1]\} + \text{const}\\ &\quad - GM_o m\{R_o^2 + h_1^2 + h_2^2 + \ell^2 + 2R_oh_1(c\theta s\phi s\psi - s\theta c\psi)\\ &\quad + 2R_oh_2(s\theta s\psi + c\theta s\phi c\psi) + 2h_1\ell s\alpha c\gamma + 2h_2\ell s\gamma\\ &\quad + 2R_o\ell(c\theta s\phi s\psi s\alpha c\gamma + s\theta s\psi s\gamma - s\theta c\psi s\alpha c\gamma\\ &\quad + c\theta s\phi c\psi s\gamma + c\theta c\phi c\alpha c\gamma)\}^{-½} \end{aligned} \tag{3}$$

where I_ξ, I_η, I_ζ are the platform pitch, roll, and yaw principal moments of inertia, r is distance between the center of Earth and the element of mass of the platform, $\mathrm{d}m$; G, M_o, M, and m are the universal gravitational constant, mass of the Earth, mass of the platform, and mass of the subsatellite, respectively. Subscripts P and S refer to the platform and subsatellite, respectively.

The equations governing the system dynamics are obtained using the classical Lagrangian formulation. After nondimensionalization and linearization about the nominal local horizontal stationkeeping orientation and with the further assumption that there is no attachment offset along the platform pitch axis (i.e., $h_2 = 0$), the linearized in-plane equations in state variable form are expressed as

$$\frac{\mathrm{d}X_{\text{in}}}{\mathrm{d}\tau} = A_{\text{in}}X_{\text{in}} + B_{\text{in}}U_{\text{in}} \tag{4a}$$

$$Y_{\text{in}} = C_{\text{in}}X_{\text{in}} \tag{4b}$$

where

$$X_{\text{in}}^T = (\epsilon\ \theta_v\ \alpha_v\ \epsilon'\ \theta_v'\ \alpha_v'); \qquad \theta_v = \theta - \theta_{eq};$$

$$\alpha_v = \alpha - \alpha_{eq}; \qquad \epsilon = (\ell - \ell_c)\ell_c$$

$$A_{\text{in}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 3 & 3\beta_1(\lambda_2 - \lambda_1 - 1)/\lambda_1 & 0 & 0 & 2 & 2\\ 0 & 3(\lambda_2 - 1)/\lambda_1 & 0 & 0 & 0 & 0\\ 0 & -3 + 3(1-\lambda_2)/\lambda_1 & -3 & -2 & 2\beta_1 & 0 \end{bmatrix} \tag{5}$$

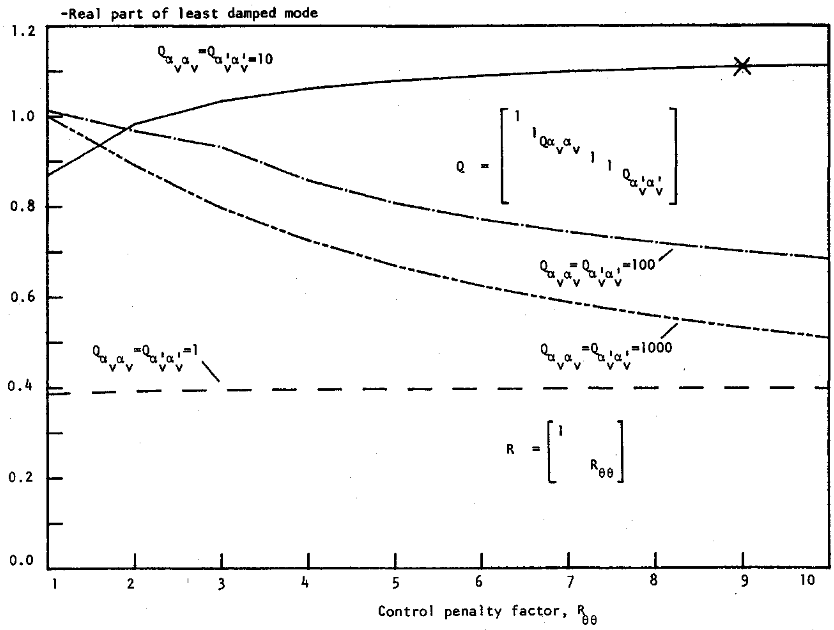

Fig. 2 Variation of least-damped mode characteristics with Q, R for in-plane motion.

$$B_{\text{in}}^T = \begin{bmatrix} 0 & 0 & 0 & 1+\beta_1^2 A/\lambda_1 & \beta_1 A/\lambda_1 & -\beta_1 A/\lambda_1 \\ 0 & 0 & 0 & \beta_1/\lambda_1 & 1/\lambda_1 & -1/\lambda_1 \end{bmatrix} \tag{6}$$

where $\tau = \omega t$, $\beta_1 = h_1/\ell_c$, $A = m\ell_c^2/I_\xi$, $\lambda_1 = I_\eta/I_\xi$, $\lambda_2 = I_\zeta/I_\xi$; ω is the platform orbital rate; and ℓ_c the tether line reference length. C_{in} is a 4×6 matrix with nonzero elements (1) appearing in C_{11}, C_{22}, C_{34}, and C_{45}.

Also, the linearized out-of-plane equations are

$$\frac{\mathrm{d}X_{\text{out}}}{\mathrm{d}\tau} = A_{\text{out}}X_{\text{out}} + B_{\text{out}}U_{\text{out}} \tag{7a}$$

$$Y_{\text{out}} = C_{\text{out}}X_{\text{out}} \tag{7b}$$

where

$$X_{\text{out}}^T = (\phi\ \psi\ \gamma\ \phi'\ \psi'\ \gamma')$$

$$A_{\text{out}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 4(\lambda_2-\lambda_1) & 3\beta_1 A & 0 & 0 & \lambda_1-\lambda_2-1 & 0 \\ 0 & (1-\lambda_1)/\lambda_2 & 3\beta_1 A/\lambda_2 & (1+\lambda_2-\lambda_1)/\lambda_2 & 0 & 0 \\ 4(1+\lambda_2-\lambda_1) & \beta_1[3A-1+(\lambda_1-1)/\lambda_2] & -4-3\beta_1^2 A/\lambda_2 & \beta_1(\lambda_1-1-\lambda_2)/\lambda_2 & \lambda_1-\lambda_2-1 & 0 \end{bmatrix} \tag{8}$$

$$B_{\text{out}}^T = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1/\lambda_2 & -\beta_1/\lambda_2 \end{bmatrix} \tag{9}$$

C_{out} is a 4×6 matrix with nonzero elements (1) appearing in C_{11}, C_{22}, C_{34}, and C_{45}.

The equilibrium approximation (to first-order) of the platform rotational and subsatellite swing angles are

$$\alpha_{\text{eq}} = -\theta_{\text{eq}} = m\ell_c h_1(I_\xi - I_\zeta - h_1^2 m)^{-1} \tag{10}$$

$$\phi_{\text{eq}} = \psi_{\text{eq}} = \gamma_{\text{eq}} = 0 \tag{11}$$

It should be noted that for the general case of attachment offset ($h_1 \neq 0$, $h_2 \neq 0$), the linearized system in-plane and out-of-plane degrees of freedom remain coupled.

For this application it is assumed that the control could be realized through appropriate modulation of the tension in the tether line and the use of momentum-type controllers for the platform pitch, roll, and yaw rotations. As to the output, for practical reasons, it is assumed that the length of the tether, and its rate, and the platform rotational angles and their rates are the only measurable components of the state.

Application of the Linear Quadratic Regulator

It can be verified that both the in-plane and out-of-plane subsystems are controllable and observable for the present example. Further analyses verified that the in-plane subsystem is controllable when the tether tension modulation is the only means of control available, but the transient dynamics may not be satisfactory. If, due to practical limitations, only the pitch angle and its rate are available as output, then the in-plane subsystem is unobservable. But, if the length and length rate of the tether are the only measurable-state components, the in-plane subsystem is still observable. It can also be verified that if the roll angle and roll angle rate are the only measurable-state components, the out-of-plane subsystem is observable. The same result is true if the yaw angle and its rate are the only measurable components.

For the numerical work in this study the following platform and subsatellite properties are considered: $M = 10{,}000.0$ kg, $m = 100.0$ kg, $I_\xi = I_\eta = 5.33 \times 10^6$ kg – m², $I_\zeta = 2I_\xi$ (for a thin square platform, each side = 80.0 m), $\omega = 1.1068 \times 10^{-3}$ rad/s (altitude = 500.0 km), $\ell_c = 100.0$ m, $h_1 = -20.0$ m. With the above system properties, the equilibrium values for the tether

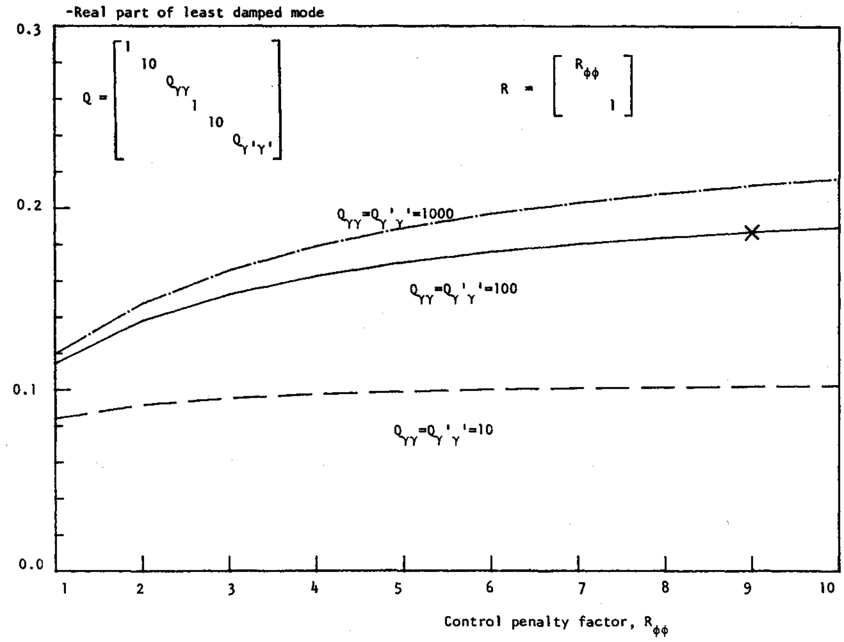

Fig. 3 Variation of least-damped mode characteristics with *Q, R* for out-of-plane motion.

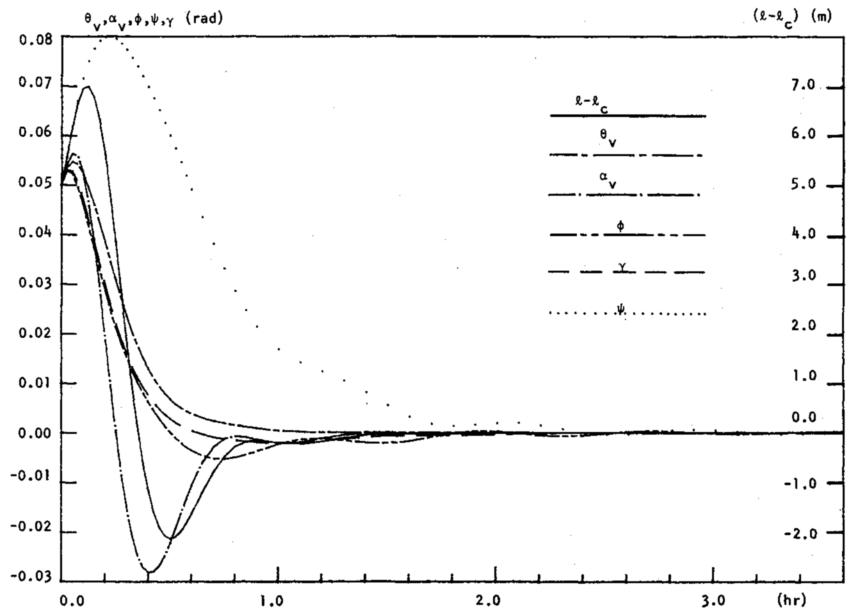

Fig. 4 Transient response of the system.

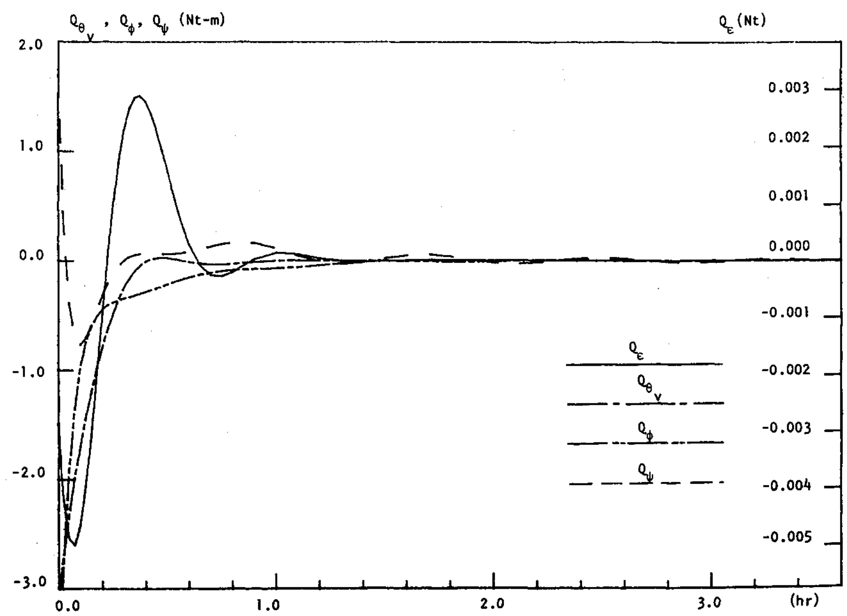

Fig. 5 Control effort required.

line in-plane swing angle and the platform pitch angle are calculated to be $-\alpha_{eq} = \theta_{eq} = -0.0372$ rad.

Figures 2 and 3 show variations of the real part of the least-damped oscillatory mode for both subsystems, with the Q and R matrices. Only by trial and error can one arrive at suitable values for Q and R that result in the desired closed-loop system response. The crosses show the optimal design point, arrived at after a series of parametric studies.

Figures 4 and 5 show the transient response of the differential length, the platform pitch, and the tether line in-plane swing angle, as well as platform yaw, roll, and tether line out-of-plane swing angle for initial conditions of 105 m in tether length, length rate 5.5×10^{-3} m/s and 0.05 rad in all the variational angles and angular rates of 5.5×10^{-5} rad/s. It is seen that it takes much more time for the out-of-plane subsystem to reach equilibrium than for the in-plane subsystem.

Concluding Remarks

The equations describing the out-of-plane motion (i.e., platform roll, yaw rotation, and tether out-of-plane swing) decouple from the in-plane motion equations (i.e., platform pitch rotation, tether in-plane swing, and motion in the direction of the tether) when the attachment point is offset only along the platform roll axis. The system is controllable with a momentum-type controller on the platform and with tension modulation on the tether line. The system is observable with tether length, length rate, platform roll and/or yaw angle, and their rate measurements. The tether attachment offset, which is the source of the system's natural coupling, is an important factor in establishing system controllability and observability. For the case of no attachment offset, rotation of the platform will not affect the subsatellite out-of-plane swing; in other words, we should consider the effect of higher-order terms or should augment the means of control, such as by placing actuators on the subsatellite to control the tether line out-of-plane swing.

Acknowledgment

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Vector Representation of Finite Rotations

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IN the following Note we restrict ourselves to real 3-vectors, right-handed Cartesian coordinate systems, and proper orthogonal transformation matrices. Euler showed that any sequence of rotations leading from one frame of reference to another can be represented by a single rotation about an axis fixed in space. The axis and sense of the rotation can be represented by a unit vector. It is a vector because the rotation axis exists independently of the coordinate systems chosen to represent the frames of reference. The angle of the rotation, like all angles, is a scalar. The product of a vector and a scalar is a vector. Therefore, the product of the unit vector and the angle of the rotation gives a vector that represents the rotation. This rotation vector can be shown to satisfy the definition of a vector: an ordered triple of real numbers that transforms from one coordinate system to another by an orthogonal transformation.

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In 1949 Laning[1] derived the algebra and calculus of such rotation vectors. That is, he found the vector operation that corresponds to the combination of two rotations into a single equivalent rotation, and the vector differential equation that corresponds to the evolution of a rotation as a function of time, given the angular velocity of one frame of reference with respect to another.

In 1950 Goldstein[2] wrote:

> Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are two such [rotation] "vectors" associated with transformations A and B. Then to qualify as vectors they must be commutative in addition...But the addition of two rotations, i.e., one rotation performed after another, it has been seen, corresponds to the product AB of the two matrices. However, matrix multiplication is not commutative...hence $\boldsymbol{A}$, $\boldsymbol{B}$ are not commutative in addition and cannot be accepted as vectors.

In 1965 Greenwood[3] wrote: "...one might think that a single rotation $\boldsymbol{\phi} = \boldsymbol{\phi}_1 + \boldsymbol{\phi}_2$ is equivalent to the rotations $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ taken in sequence. But we have seen that rigid body rotations are not commutative, in general, whereas vector addition is commutative. Therefore a general rigid-body rotation is not a vector quantity."

In 1970 Meirovitch[4] wrote: "...in general, matrix products are not commutative...Hence, *finite angles of rotation cannot be represented by vectors.*"

In 1986 Hughes[5] wrote: "because rotation matrices do not commute in multiplication, the search for an angular displacement 'vector' is futile."

These authors use the following reasoning: the combination of rotations does not commute; the addition of vectors commutes; therefore rotations cannot be represented by vectors. Of course, the correct conclusion is that the combination of rotations cannot be represented by the addition of vectors. (The addition of two rotation vectors does, in fact, commute, but it does not, in general, represent the combination of the two corresponding rotations. Laning's vector operation does not commute and does represent the combination of the two rotations.)

It is pedagogically important to understand that whether or not an ordered triple of real numbers is a vector is determined by its transformation properties, not by whether particular vector operations on it are physically meaningful. The operations follow from the definition of a vector and the laws of real number arithmetic, not vice versa. The operations we use are selected for their physical usefulness from the set of all possible operations on vectors. Whenever we apply vectors to a new analysis, we may need to invent a new vector operation as Laning did.

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